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# Positive periodic solutions for a nonlinear difference system via a continution theorem

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**Abstract.** Based on a continuation theorem of Mawhin, the existence of a positive periodic solution for a nonlinear difference system is studied.

**Keywords:** Nonlinear difference system, positive periodic solution, continution theorem.

**Mathematical subject classification:** 39A11.

## 1 Introduction

In [1], we explained that scalar difference equations of the form

$$y_{n+1} = y_n \exp\{f(n, y_n, y_{n-1}, ..., y_{n-k})\}, \ n \in Z = \{0, \pm 1, \pm 2, ...\}, \qquad (1)$$

where $f = f(t, u_0, u_1, ..., u_k)$ is a real continuous function defined on $R^{k+2}$ such that

$$f(t + \omega, u_0, ..., u_k) = f(t, u_0, ..., u_k), \ (t, u_0, ..., u_k) \in R^{k+2},$$

and $\omega$ is a positive integer, are of interest since they include well known equations such as

$$y_{n+1} = y_n \exp\left\{\frac{\mu(1 - y_n)}{K}\right\}, \ K > 0,$$

and they are intimately related to delay differential equations with piecewise constant independent arguments [2]:

$$y'(t) = y(t)\ f([t], y([t]), y([t-1]), y([t-2]), ..., y([t-k])), \ t \in R.$$

We also show that continuation theorems can be used to show existence of periodic solutions of these equations.

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Note that in the above equations, only one time dependent variable $y_t$ is involved. In real problems, multiple time dependent variables may interact, and therefore it is natural to study systems of difference equations.

In this paper, we consider one such system of the form

$$y_i^{(n+1)} = y_i^{(n)} \exp\left( r_i^{(n)} - \sum_{j=1}^{k} a_{ij}^{(n)} y_j^{(n)} - \sum_{j=1}^{k} b_{ij}^{(n)} y_j^{\left(n-\tau_{ij}^{(n)}\right)} \right),$$

$$i \in \{1, ..., k\}, n \in Z, \tag{2}$$

where

$$r_i = \left\{ r_i^{(n)} \right\}_{n \in Z}, \quad a_{ij} = \left\{ a_{ij}^{(n)} \right\}_{n \in Z}, \quad b_{ij} = \left\{ b_{ij}^{(n)} \right\}_{n \in Z} \quad \text{and} \quad \tau_{ij} = \left\{ \tau_{ij}^{(n)} \right\}_{n \in Z},$$

are real $\omega$-periodic sequences such that

$$\begin{aligned} r_i^{(n)} &= r_i^{(n+\omega)}, \ n \in Z \\ a_{ij}^{(n)} &= a_{ij}^{(n+\omega)}, \ n \in Z \\ b_{ij}^{(n)} &= b_{ij}^{(n+\omega)}, \ n \in Z \\ \tau_{ij}^{(n)} &= \tau_{ij}^{(n+\omega)}, \ n \in Z \end{aligned}$$

for $i, j \in \{1, ..., k\}$. We assume further that

$$a_{ij}^{(n)}, b_{ij}^{(n)} \geqslant 0, \ i, j \in \{1, ..., k\}; n \in Z,$$

$$\sum_{0 \le n \le \omega - 1} r_i^{(n)} > 0, \ i \in \{1, ..., k\},$$

and

$$\sum_{0 \le n \le \omega - 1} \left( a_{ii}^{(n)} + b_{ii}^{(n)} \right) \neq 0, \ i \in \{1, ..., k\}.$$

The number $\omega$ is a positive integer as before.

A solution of (2) is a real vector sequence of the form $y = \left\{ y^{(n)} \right\}_{n \in Z}$ where $y^{(n)} = \left( y_1^{(n)}, y_2^{(n)}, ..., y_k^{(n)} \right)^{\dagger}$ which renders (2) into an identity after substitution. As in [1], we are concerned with the existence of positive solutions which are $\omega$-periodic, that is, solutions that satisfy $y^{(n+\omega)} = y^{(n)}$ for $n \in Z$ and $y_i^{(n)} > 0$ for $n \in Z$ and $i \in \{1, ..., k\}$.

Our system (2) can be used to describe multispecies ecological competition systems or multi-nation competition models. The analogous problem for differential systems has been treated by Smith [4], Cushing [5], Zanolin [6], Fan and

Wang [7] and others. In particular, in [7], the authors study differential systems of the form

$$y_i'(t) = y_i(t)\left(r_i(t) - \sum_{j=1}^{k} a_{ij}(t)y_j(t) - \sum_{j=1}^{k} b_{ij}(t)y_j(t-\tau_{ij})\right), \ i = 1, 2, ..., k.$$

As for our system, we can also show that it is related to differential systems with piecewise constant independent arguments of the form

$$y_i'(t) = y_i(t)\left(r_i([t]) - \sum_{j=1}^{k} a_{ij}([t])\, y_j(n) - \sum_{j=1}^{k} b_{ij}([t])\, y_j\left([t] - \tau_{ij}([t])\right)\right),$$
$$i \in \{1, ..., k\}, t \in R, \tag{3}$$

where $[x]$ is the greatest-integer function, $r_i(t)$, $a_{ij}(t)$ and $b_{ij}(t)$ are real continuous $\omega$-periodic functions defined on $R$. Indeed, once the existence of a positive $\omega$-periodic solution of (2) can be demonstrated, we may then make immediate statements about the existence of positive $\omega$-periodic solutions of (3). The proof of our assertion is not much different from that of Theorem 1 in [1], and hence is not included here.

As in [1], we will invoke a continuation theorem of Mawhin for obtaining periodic solutions of (2). For the sake of easy reference, we briefly describe this result here. Let $X$ and $Y$ be two Banach spaces and $L : \mathrm{Dom}L \subset X \to Y$ is a linear mapping and $N : X \to Y$ a continuous mapping. The mapping $L$ will be called a Fredholm mapping of index zero if $\dim \mathrm{Ker}L = \mathrm{codim}\ \mathrm{Im}L < +\infty$, and $\mathrm{Im}L$ is closed in $Y$. If $L$ is a Fredholm mapping of index zero, there exist continuous projectors $P : X \to X$ and $Q : Y \to Y$ such that $\mathrm{Im}P = \mathrm{Ker}L$ and $\mathrm{Im}L = \mathrm{Ker}Q = \mathrm{Im}(I - Q)$. It follows that $L_{|\mathrm{Dom}L\cap\mathrm{Ker}P} : (I - P)X \to \mathrm{Im}L$ has an inverse which will be denoted by $K_P$. If $\Omega$ is an open and bounded subset of $X$, the mapping $N$ will be called $L$-compact on $\bar{\Omega}$ if $QN\left(\bar{\Omega}\right)$ is bounded and $\overline{K_P(I-Q)N\left(\bar{\Omega}\right)}$ is compact. Since $\mathrm{Im}Q$ is isomorphic to $\mathrm{Ker}L$ there exist an isomorphism $J : \mathrm{Im}Q \to \mathrm{Ker}L$.

**Theorem A (Mawhin's continuation theorem [1]).** *Let $L$ be a Fredholm mapping of index zero, and let $N$ be $L$-compact on $\bar{\Omega}$. Suppose*

(i) *for each* $\lambda \in (0, 1)$, $x \in \partial\Omega$, $Lx \neq \lambda Nx$; *and*

(ii) *for each* $x \in \partial\Omega \cap \mathrm{Ker}L$, $QNx \neq 0$ *and* $\deg(JQN, \Omega \cap \mathrm{Ker}L, 0) \neq 0$.

*Then the equation $Lx = Nx$ has at least one solution in $\bar{\Omega} \cap \mathrm{dom}L$.*

We recall the useful nonstandard "summation" operation [1] for any real sequence $\{u^{(n)}\}_{n\in Z}$:

$$\bigoplus_{n=\gamma}^{\beta} u^{(n)} = \begin{cases} \sum_{n=\gamma}^{\beta} u^{(n)}, & \gamma \le \beta \\ 0, & \beta = \gamma - 1 \\ -\sum_{n=\beta+1}^{\gamma-1} u^{(n)}, & \beta < \gamma - 1 \end{cases}.$$

As usual, the forward difference is defined by $\Delta u^{(k)} = u^{(k+1)} - u^{(k)}$.

We will also employ the following notations for the 'time' averages:

$$\bar{r}_i = \frac{1}{\omega} \sum_{0 \le n \le \omega-1} r_i^{(n)},$$

$$\overline{R}_i = \frac{1}{\omega} \sum_{0 \le n \le \omega-1} \left| r_i^{(n)} \right|,$$

$$\bar{a}_{ij} = \frac{1}{\omega} \sum_{0 \le n \le \omega-1} a_{ij}^{(n)},$$

$$\bar{b}_{ij} = \frac{1}{\omega} \sum_{0 \le n \le \omega-1} b_{ij}^{(n)}.$$

## 2 Existence Criteria

The main result of our paper is the following.

**Theorem 1.** *Suppose the following set of conditions hold:*

(i) *for each $i \in \{1, ..., k\}$, $\bar{r}_i > 0$,*

(ii) *for $i, j \in \{1, ..., k\}$, the inverse of the matrix $\left(\bar{a}_{ij} + \bar{b}_{ij}\right)_{k\times k}$ exists and all its components are positive, and*

(iii) *for each $i \in \{1, ..., k\}$,*

$$\bar{r}_i > \sum_{1 \le j \le k, j \ne i} \left(\bar{a}_{ij} + \bar{b}_{ij}\right) \frac{\bar{r}_j}{\bar{a}_{jj} + \bar{b}_{jj}} \exp\left(\frac{1}{2}\left(\overline{R}_j + \bar{r}_j\right)\omega\right).$$

*Then (2) has a positive $\omega$-periodic solution.*

In order to provide a proof, we proceed in a manner similar to that of Theorem 1 in [1]. However, there are sufficient difference to warrant some details in the following discussions. We first note that if

$$x = \left\{ \left( x_1^{(n)}, x_2^{(n)}, ..., x_k^{(n)} \right)^{\dagger} \right\}_{n \in Z}$$

is a $\omega$-periodic solution of the following system

$$x_i^{(n)} = x_i^{(0)} + \bigoplus_{s=0}^{n-1} \left( r_i^{(s)} - \sum_{j=1}^{k} a_{ij}^{(s)} \exp\left(x_j^{(s)}\right) - \sum_{j=1}^{k} b_{ij}^{(s)} \exp x_j^{\left(s-\tau_{ij}^{(s)}\right)} \right),$$
$$i \in \{1, ..., k\}, n \in Z, \tag{4}$$

then we can easily check that

$$y = \left\{ \left( y_1^{(n)}, y_2^{(n)}, ..., y_k^{(n)} \right)^{\dagger} \right\}_{n \in Z} = \left\{ \left( e^{x_1^{(n)}}, e^{x_2^{(n)}}, ..., e^{x_k^{(n)}} \right)^{\dagger} \right\}_{n \in Z}$$

is a positive $\omega$-periodic solution of (1).

We will therefore seek an $\omega$-periodic solution of (4). Let $X_\omega$ be the Banach space of all real vector $\omega$-periodic sequences of the form $x = \{x^{(n)}\}_{n \in Z}$ where $x^{(n)} = \left( x_1^{(n)}, x_2^{(n)}, ..., x_k^{(n)} \right)^{\dagger}$ and endowed with the usual linear structure as well as the norm

$$\|x\|_1 = \left( \sum_{1 \le i \le k} \left( \max_{0 \le n \le \omega - 1} \left| x_i^{(n)} \right| \right)^2 \right)^{\frac{1}{2}}.$$

Let $Y_\omega$ be the Banach space of all real sequences of the form

$$y = \{y^{(n)}\}_{n \in Z} = \left\{ n\alpha + h^{(n)} \right\}_{n \in Z}$$

such that $y^{(0)} = 0$, where $\alpha = (\alpha_1, ..., \alpha_k)^{\dagger} \in R^k$ and $\{h^{(n)}\}_{n \in Z} \in X_\omega$, and endowed with the usual linear structure as well as the norm $\|y\|_2 = |\alpha| + \|h\|_1$, here $|\alpha| = \left( \sum_{1 \le i \le k} \alpha_i^2 \right)^{\frac{1}{2}}$. Let the zero element of $X_\omega$ and $Y_\omega$ be denoted by $\theta_1$ and $\theta_2$ respectively.

Define the mappings $L : X_\omega \to Y_\omega$ and $N : X_\omega \to Y_\omega$ respectively by

$$(Lx)^{(n)} = x^{(n)} - x^{(0)}, \ n \in Z. \tag{5}$$

and

$$(Nx)_i^{(n)} = \bigoplus_{s=0}^{n-1} \left( r_i^{(s)} - \sum_{j=1}^{k} a_{ij}^{(s)} \exp x_j^{(s)} - \sum_{j=1}^{k} b_{ij}^{(s)} \exp x_j^{\left(s-\tau_{ij}^{(s)}\right)} \right),$$

$$i \in \{1, ..., k\}, n \in Z. \tag{6}$$

Let

$$\begin{aligned} \bar{h}_i^{(n)} = & \bigoplus_{s=0}^{n-1} \left( r_i^{(s)} - \sum_{j=1}^{k} a_{ij}^{(s)} \exp x_j^{(s)} - \sum_{j=1}^{k} b_{ij}^{(s)} \exp \left( x_j^{\left(s-\tau_{ij}^{(s)}\right)} \right) \right) \\ & - \frac{n}{\omega} \bigoplus_{s=0}^{\omega-1} \left( r_i^{(s)} - \sum_{j=1}^{k} a_{ij}^{(s)} \exp x_j^{(s)} - \sum_{j=1}^{k} b_{ij}^{(s)} \exp \left( x_j^{\left(s-\tau_{ij}^{(s)}\right)} \right) \right) \end{aligned} \tag{7}$$

for $i = 1, ..., k$ and $n \in Z$.

Since $\bar{h} = \{\bar{h}^{(n)}\}_{n \in Z} \in X_\omega$ and $\bar{h}^{(0)} = \theta_1$, $N$ is a well-defined operator from $X_\omega$ to $Y_\omega$. On the other hand, direct calculation shows that $\mathrm{Ker} L = \{x \in X_\omega \mid x^{(n)} = x^{(0)}, n \in Z, x^{(0)} \in R^k\}$ and $\mathrm{Im} L = X_\omega \cap Y_\omega$. Let us define $P : X_\omega \to X_\omega$ and $Q : Y_\omega \to Y_\omega$ respectively by

$$(Px)^{(n)} = x^{(0)}, \quad n \in Z, \tag{8}$$

for $x = \{x^{(n)}\}_{n \in Z} \in$ and

$$(Qy)^{(n)} = n\alpha \tag{9}$$

for $y = \{n\alpha + h^{(n)}\}_{n \in Z} \in Y_\omega$. The operators $P$ and $Q$ are projections and $X_\omega = \mathrm{Ker} P \oplus \mathrm{Ker} L$, $Y_\omega = \mathrm{Im} L \oplus \mathrm{Im} Q$. It is easy to see that $\dim \mathrm{Ker} L = k = \dim \mathrm{Im} Q = \mathrm{codim} \mathrm{Im} L$, and

$$\mathrm{Im} L = \{y \in X_\omega \mid y(0) = 0\} \subset Y_\omega.$$

It follows that $\mathrm{Im} L$ is closed in $Y_\omega$. Thus the following Lemma is true.

**Lemma 1.** *The mapping $L$ defined by (5) is a Fredholm mapping of index zero.*

**Lemma 2.** *Let $L$ and $N$ defined by (5) and (6) respectively. Suppose $\Omega$ is an open and bounded subset of $X_\omega$. Then $N$ is $L$-compact on $\overline{\Omega}$.*

**Proof.** From (6), (7) and (9), we see that for any $x = \{x^{(n)}\}_{n\in Z} \in X_\omega$,

$$(QNx)_i^{(n)} = \frac{n}{\omega} \bigoplus_{s=0}^{\omega-1} \left( r_i^{(s)} - \sum_{j=1}^{k} a_{ij}^{(s)} \exp x_j^{(s)} - \sum_{j=1}^{k} b_{ij}^{(s)} \exp \left( x_j^{\left(s-\tau_{ij}^{(s)}\right)} \right) \right),$$

$$i \in \{1, 2, .., k\}, n \in Z. \tag{10}$$

We denote the inverse of the mapping $L \mid_{\mathrm{Dom}L\cap \mathrm{Ker}P}: (I - P)\, X \to \mathrm{Im}L$ by $K_P$. Direct calculation leads to

$$(K_P(I-Q)Nx)_i^{(n)} = \bigoplus_{s=0}^{n-1} \left( r_i^{(s)} - \sum_{j=1}^{k} a_{ij}^{(s)} \exp x_j^{(s)} - \sum_{j=1}^{k} b_{ij}^{(s)} \exp x_j^{\left(s-\tau_{ij}^{(s)}\right)} \right)$$

$$-\frac{n}{\omega} \bigoplus_{s=0}^{\omega-1} \left( r_i^{(s)} - \sum_{j=1}^{k} a_{ij}^{(s)} \exp x_j^{(s)} - \sum_{j=1}^{k} b_{ij}^{(s)} \exp x_j^{\left(s-\tau_{ij}^{(s)}\right)} \right). \tag{11}$$

It is easy to see that $QN$ and $K_P(I - Q)N$ are continuous on $X_\omega$ and takes bounded sets into bounded sets respectively. Since the Banach space $X_\omega$ is finite dimensional, $N$ is $L$-compact on $\overline{\Omega}$. The proof is complete. □

Let $l_\omega$, where $\omega \geqslant 2$ is positive number, be the space of all real $\omega$-periodic sequences of the form $u = \left\{u^{(n)}\right\}_{n\in Z}$.

**Lemma 3.** *If* $u = \left\{u^{(n)}\right\}_{n\in Z} \in l_\omega$, *then*

$$\max_{0\le s,i\le \omega-1} \left|u^{(s)} - u^{(i)}\right| \le \frac{1}{2} \sum_{k=0}^{\omega-1} \left|\Delta u^{(k)}\right|, \tag{12}$$

*where the constant factor* $1/2$ *is the best possible.*

**Proof.** Let $u = \left\{u^{(n)}\right\}_{n\in Z} \in l_\omega$ and $s, i \in \{0, 1, ..., \omega - 1\}$. Without loss of any generality, let $s \in \{i + 1, ..., i + \omega - 1\}$, we have

$$u^{(s)} = u^{(i)} + \sum_{k=i}^{s-1} \Delta u^{(k)} \tag{13}$$

and

$$u^{(i)} = u^{(i+\omega)} = u^{(s)} + \sum_{k=s}^{i+\omega-1} \Delta u^{(k)}. \tag{14}$$

From (13) and (14), we see that for any $s \in \{i, i+1, \ldots, i+\omega-1\}$,

$$2u^{(s)} = 2u^{(i)} + \sum_{k=i}^{s-1} \Delta u^{(k)} - \sum_{k=s}^{i+\omega-1} \Delta u^{(k)}, \tag{15}$$

that is

$$u^{(s)} = u^{(i)} + \frac{1}{2}\left\{\sum_{k=i}^{s-1} \Delta u^{(k)} - \sum_{k=s}^{i+\omega-1} \Delta u^{(k)}\right\}. \tag{16}$$

Thus for any $s \in \{i, i+1, \ldots, i+\omega-1\}$,

$$\left|u^{(s)} - u^{(i)}\right| \le \frac{1}{2}\sum_{k=i}^{i+\omega-1} \left|\Delta u^{(k)}\right| = \frac{1}{2}\sum_{k=0}^{\omega-1}\left|\Delta u^{(k)}\right|, \tag{17}$$

so that

$$\max_{0\le s,i\le\omega-1}\left|u^{(s)} - u^{(i)}\right| \le \frac{1}{2}\sum_{k=0}^{\omega-1}\left|\Delta u^{(k)}\right|. \tag{18}$$

Now we assert that if $\beta$ is a constant and $\beta < 1/2$, then there are $u = \left\{u^{(n)}\right\}_{n\in Z} \in l_\omega$ and such that

$$\max_{0\le s,i\le\omega-1}\left|u^{(s)} - u^{(i)}\right| > \beta\sum_{k=0}^{\omega-1}\left|\Delta u^{(k)}\right|. \tag{19}$$

Indeed, if we let $u^{(n)} = j$ for $n = k\omega + j$, $k \in Z$ and $j = 0, 1, ..., \omega - 1$, then $\max_{0\le s,i\le\omega-1}\left|u^{(s)} - u^{(i)}\right| = \omega - 1$ and

$$\Delta u^{(n)} = \begin{cases} 1, & n = 0, 1, .., \omega-2 \\ -(\omega-1), & n = \omega - 1 \end{cases}, \tag{20}$$

and

$$\beta\sum_{k=0}^{\omega-1}\left|\Delta u^{(k)}\right| = 2\beta(\omega-1) < \max_{0\le s,i\le\omega-1}\left|u^{(s)} - u^{(i)}\right|$$

as required. This shows that the constant $1/2$ in (12) is the best possible. The proof is complete. □

Now, we consider the following system

$$x_i^{(n)} - x_i^{(0)} = \lambda \bigoplus_{s=0}^{n-1}\left(r_i^{(s)} - \sum_{j=1}^{k} a_{ij}^{(s)} \exp x_j^{(s)} - \sum_{j=1}^{k} b_{ij}^{(s)} \exp\left(x_j^{\left(s-\tau_{ij}^{(s)}\right)}\right)\right),$$

$$i \in \{1, ..., k\}, n \in Z, \tag{21}$$

where $\lambda \in (0, 1)$.

**Lemma 4.** *Suppose the condition (iii) in Theorem 1 holds. Then there exist positive constants* $H_1, ..., H_k$ *such that for any solution* $x = \{x^{(n)}\}_{n\in Z} = \left\{\left(x_1^{(n)}, ..., x_k^{(n)}\right)^\dagger\right\}_{n\in Z} \in X_\omega$ *of (21), we have the following inequalities*

$$\max_{0\le n\le \omega-1}\left|x_i^{(n)}\right| \le H_i, \ \ i \in \{1, ..., k\}. \tag{22}$$

**Proof.** Let $x = \{x^{(n)}\}_{n\in Z}$ be a $\omega$-periodic solution of (21). Then

$$\bigoplus_{s=0}^{\omega-1}\left(r_i^{(s)} - \sum_{j=1}^{k} a_{ij}^{(s)} \exp x_j^{(s)} - \sum_{j=1}^{k} b_{ij}^{(s)} \exp x_j^{\left(s-\tau_{ij}^{(s)}\right)}\right) = 0, \ \ i \in \{1, ..., k\}. \tag{23}$$

It leads to

$$\bigoplus_{s=0}^{\omega-1}\left(\sum_{j=1}^{k} a_{ij}^{(s)} \exp x_j^{(s)} + \sum_{j=1}^{k} b_{ij}^{(s)} \exp x_j^{\left(s-\tau_{ij}^{(s)}\right)}\right) = \omega\overline{r_i}. \tag{24}$$

From (21), we have

$$\Delta x_i^{(n)} = \lambda\left(r_i^{(n)} - \sum_{j=1}^{k} a_{ij}^{(n)} \exp x_j^{(n)} - \sum_{j=1}^{k} b_{ij}^{(n)} \exp\left(x_j^{\left(n-\tau_{ij}^{(n)}\right)}\right)\right),$$

$$i \in \{1, ..., k\}, n \in Z. \tag{25}$$

By (24) and (25), we see that

$$\begin{aligned}\bigoplus_{s=0}^{\omega-1}\left|\Delta x_i^{(s)}\right| &\le \bigoplus_{s=0}^{\omega-1}\left(\left|r_i^{(s)}\right| + \sum_{j=1}^{k} a_{ij}^{(s)} \exp x_j^{(s)} + \sum_{j=1}^{k} b_{ij}^{(s)} \exp x_j^{\left(s-\tau_{ij}^{(s)}\right)}\right)\\ &= \bigoplus_{s=0}^{\omega-1}\left|r_i^{(s)}\right| + \bigoplus_{s=0}^{\omega-1}\left(\sum_{j=1}^{k} a_{ij}^{(s)} \exp\left(x_j^{(s)}\right) + \sum_{j=1}^{k} b_{ij}^{(s)} \exp x_j^{\left(s-\tau_{ij}^{(s)}\right)}\right)\\ &= \left(\overline{R}_i + \overline{r}_i\right)\omega.\end{aligned} \tag{26}$$

Let $x_i^{(\mu_i)} = \max_{0\le n\le \omega-1} x_i^{(n)}$ and $x_i^{(\nu_i)} = \min_{0\le n\le \omega-1} x_i^{(n)}$, where $0 \le \mu_i, \nu_i \le \omega - 1$. From (24), we have

$$\begin{aligned}\omega\bar{r}_i &\geqslant \bigoplus_{s=0}^{\omega-1}\left(\sum_{j=1}^{k} a_{ij}^{(s)} \exp x_j^{(\nu_j)} + \sum_{j=1}^{k} b_{ij}^{(s)} \exp x_j^{(\nu_j)}\right)\\ &= \sum_{j=1}^{k}\left(\bar{a}_{ij} + \bar{b}_{ij}\right)\omega \exp x_j^{(\nu_j)} \\ &\geqslant \left(\bar{a}_{ii} + \bar{b}\right)\omega \exp x_i^{(\nu_i)},\end{aligned} \tag{27}$$

that is,

$$x_i^{(\nu_i)} \le \ln\left\{\frac{\bar{r}_i}{\bar{a}_{ii} + \bar{b}_{ii}}\right\}. \tag{28}$$

In view of Lemma 3, (26) and (28), we see that for any $n = 0, 1, \ldots, \omega - 1$,

$$x_i^{(n)} \le x_i^{(\nu_i)} + \frac{1}{2}\sum_{k=0}^{\omega-1}\left|\Delta x_i^{(k)}\right| \le B_i, \tag{29}$$

where

$$B_i = \ln\left\{\frac{\bar{r}_i}{\bar{a}_{ii} + \bar{b}_{ii}}\right\} + \frac{1}{2}\left(\bar{R}_i + \bar{r}_i\right)\omega. \tag{30}$$

Furthermore, from (24), we have

$$\begin{aligned}\omega\bar{r}_i &\le \bigoplus_{s=0}^{\omega-1}\left(\sum_{j=1}^{k} a_{ij}^{(s)} \exp x_j^{(\mu_j)} + \sum_{j=1}^{k} b_{ij}^{(s)} \exp x_j^{(\mu_j)}\right)\\ &= \sum_{j=1}^{k}\left(\bar{a}_{ij} + \bar{b}_{ij}\right)\omega \exp x_j^{(\mu_j)}.\end{aligned} \tag{31}$$

By (29) and (31), we see that

$$\begin{aligned}&\left(\bar{a}_{ii} + \bar{b}_{ii}\right)\exp x_i^{(\mu_i)}\\ &\geqslant \bar{r}_i - \sum_{1\le j\le k, j\ne i}\left(\bar{a}_{ij} + \bar{b}_{ij}\right)\exp x_j^{(\mu_j)}\\ &\geqslant \bar{r}_i - \sum_{1\le j\le k, j\ne i}\left(\bar{a}_{ij} + \bar{b}_{ij}\right)\frac{\bar{r}_j}{\bar{a}_{jj} + \bar{b}_{jj}}\exp\left(\frac{1}{2}\left(\bar{R}_j + \bar{r}_j\right)\omega\right),\end{aligned} \tag{32}$$

that is

$$x_i^{(\mu_i)} \geqslant C_i, \tag{33}$$

where

$$C_i = \ln \left\{ \frac{\overline{r}_i - \sum_{1 \le j \le k, j \ne i} \left( \overline{a}_{ij} + \overline{b}_{ij} \right) \frac{\overline{r}_j}{\overline{a}_{jj} + \overline{b}_{jj}} \exp\left( \frac{1}{2} \left( \overline{R}_j + \overline{r}_j \right) \omega \right)}{\overline{a}_{ii} + \overline{b}_{ii}} \right\}. \tag{34}$$

In view of Lemma 3, (26) and (33), we see that for any $n = 0, 1, ..., \omega - 1$,

$$x_i^{(n)} \geqslant x_i^{(\mu_i)} - \frac{1}{2} \sum_{k=0}^{\omega-1} \left| \Delta x_i^{(k)} \right| \geqslant C_i - \frac{1}{2} \left( \overline{R_i} + \overline{r_i} \right) \omega. \tag{35}$$

From (29) and (35), we have

$$\max_{0 \le n \le \omega - 1} \left| x_i^{(n)} \right| \le H_{i,} \tag{36}$$

where $H_i = \max \left\{ |B_i|, \left| C_i - \frac{1}{2} \left( \overline{R_i} + \overline{r_i} \right) \omega \right| \right\} + 1$. The proof is complete. □

**Proof of Theorem 1.** Let $L$, $N$, $P$ and $Q$ be defined by (5), (6), (8) and (9) respectively. By conditions (i) and (ii), we know that the linear system of the form

$$\overline{r}_i - \sum_{j=1,}^{k} \left( \overline{a}_{ij} + \overline{b}_{ij} \right) v_j = 0, \; i \in \{1, ..., k\}, \tag{37}$$

has the unique solution $v^* = \left( v_1^*, v_2^*, ..., v_k^* \right)^\dagger$ and $v_i^* > 0$ for $i \in \{1, ..., k\}$. Pick $M$ such that

$$\left( \sum_{i=1}^{k} \left( \ln v_i^* \right)^2 \right)^{\frac{1}{2}} < M. \tag{38}$$

From Lemma 4, we know there exist positive constants $H_1, ..., H_k$ such that for any solution $x = \{x^{(n)}\}_{n \in Z} = \left\{ \left( x_1^{(n)}, ..., x_k^{(n)} \right)^\dagger \right\}_{n \in Z} \in X_\omega$ of (21), we have the following inequalities

$$\max_{0 \le n \le \omega - 1} \left| x_i^{(n)} \right| \le H_i, \; i = 1, ..., k. \tag{39}$$

Let $H = \left( \sum_{i=1}^{k} H_i^2 \right)^{\frac{1}{2}} + M$. Then $\|x\|_1 < H$. Set

$$\Omega = \{ x \in X_\omega | \; \|x\|_1 < H \}.$$

It is easy to see that $\Omega$ is an open and bounded subset of $X_\omega$ and for each $\lambda \in (0, 1)$ and $x \in \partial\Omega$, $Lx \neq \lambda Nx$. Furthermore, in view of Lemma 1 and Lemma 2, $L$ is a Fredholm mapping of index zero and $N$ is $L$-compact on $\overline{\Omega}$. Noting that $H > \left(\sum_{i=1}^{k} H_i^2\right)^{\frac{1}{2}}$, by Lemma 4, for each $\lambda \in (0, 1)$ and $x \in \partial\Omega$, $Lx \neq \lambda Nx$. Next note that a vector sequence $x = \{x^{(n)}\}_{n\in Z} \in \partial\Omega \cap \mathrm{Ker}L$ must be a constant vector and $\|x\|_1 = H > M$. Hence

$$\|QNx\|_2 = \left\{\sum_{i=1}^{k}\left(\left(\overline{r}_i - \sum_{j=1}^{k}\left(\overline{a}_{ij} + \overline{b}_{ij}\right)\exp x_j\right)\right)^2\right\}^{\frac{1}{2}} \neq 0.$$

So

$$QNx \neq \theta_2.$$

The isomorphism $J\colon \mathrm{Im}\,Q \to \mathrm{Ker}L$ is defined by $JQy = \alpha$ for $y = \{n\alpha + h^{(n)}\}_{n\in Z} \in Y_\omega$. Then

$$\begin{aligned}(JQNx)_i^{(n)} &= \frac{1}{\omega}\bigoplus_{s=0}^{\omega-1}\left(r_i^{(s)} - \sum_{j=1}^{k} a_{ij}^{(s)}\exp x_j^{(s)} - \sum_{j=1}^{k} b_{ij}^{(s)}\exp x_j^{\left(s-\tau_{ij}^{(s)}\right)}\right)\\ &= \overline{r}_i - \sum_{j=1}^{k}\left(\overline{a}_{ij} + \overline{b}_{ij}\right)\exp x_j,\end{aligned} \tag{40}$$

for $n \in Z$ and $i \in \{1, ..., k\}$. Since (37) has the unique solution $v^* = \left(v_1^*, v_2^*, ..., v_k^*\right)^\dagger$ with positive components and such that (38) is satisfied, we see that the system

$$\overline{r}_i - \sum_{j=1}^{k}\left(\overline{a}_{ij} + \overline{b}_{ij}\right)\exp\left(x_j\right) = 0,\ i \in \{1, ..., k\} \tag{41}$$

has a unique solution $\overline{x} = \left(x_1^*, x_2^*, ..., x_k^*\right)^\dagger$ in $\Omega \cap \mathrm{Ker}\,L$, so that from the condition (ii) we have

$$\deg\left(JQNx, \Omega \cap \mathrm{Ker}\,L, \theta_1\right) = \text{ sign } \det \Upsilon_{JQN}\left(\overline{x}\right) \neq 0.$$

where $\Upsilon_{JQN}\left(\overline{x}\right)$ is the Jacobi matrix of $JQN$ at $\overline{x}$. By Theorem A, we see that equation $Lx = Nx$ has at least one solution in $\overline{\Omega} \cap \mathrm{Dom}L$. In other words, (4) has a $\omega$-periodic solution $x = \{x^{(n)}\}_{n\in Z}$, and hence $\left\{\left(e^{x_1^{(n)}}, ..., e^{x_k^{(n)}}\right)\right\}_{n\in Z}$ is a positive $\omega$-periodic solution of (2). The proof is complete. □

We remark that by the relationship that exists between (2) and (3), under the same assumption of Theorem 1, system (3) has a positive $\omega$-periodic solution.

We now illustrate our main result by considering the following system

$$\begin{cases} y_1^{(n+1)} = y_1^{(n)} \exp\left(r_1^{(n)} - a_{11}^{(n)} y_1^{(n)} - b_{12}^{(n)} y_2^{\left(n-\tau_{12}^{(n)}\right)}\right) & , \\ y_2^{(n+1)} = y_2^{(n)} \exp\left(r_2^{(n)} - a_{22}^{(n)} y_2^{(n)} - b_{21}^{(n)} y_1^{\left(n-\tau_{21}^{(n)}\right)}\right) & , \end{cases}$$

where $r_i, b_{ij}, a_{ii}$ and $\tau_{ij}$ for $i, j \in \{1, 2\}$ are 2-periodic sequences and

$$r_1^{(0)} = 0,\ r_1^{(1)} = 1,\ r_2^{(0)} = 1,\ r_2^{(1)} = 0,\ a_{11}^{(0)} = 1/3,\ a_{11}^{(1)} = 2/3,\ a_{22}^{(0)} = 2/3,$$

$$a_{22}^{(1)} = 1/3,\ b_{12}^{(0)} = 1/6e,\ b_{12}^{(1)} = 1/4e,\ b_{21}^{(0)} = 1/5e,\ b_{21}^{(1)} = 1/7e.$$

It is easily verified from Theorem 1 that it has a positive 2-periodic solution.

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